Catastrophe Derivatives and Reinsurance Contracts: An Incomplete Markets Approach

Stylianos Perrakis and Ali Boloorforoosh*

*The authors are, respectively, RBC Distinguished Professor of Financial Derivatives and PhD Candidate in Finance at the John Molson School of Business, Concordia University. Emails: sperrakis@jmsb.concordia.ca and aboloor@jmsb.concordia.ca. They wish to thank the Social Sciences and Humanities Research Council (SSHRC), the Institut de Finance Mathmatique (IFM2), the Montreal Institute of Structured Finance and Derivatives (IFSID), and the RBC Distinguished Professorship in Financial Derivatives for financial support.
Abstract

We present a theoretical methodology based on stochastic dominance for the pricing of catastrophe (CAT) derivatives with non-convex payoffs given the price of a CAT indexed futures contract. We do not assume a fully diversifiable CAT event risk, nor do we assume knowledge of the martingale probability measure beyond the futures price. We derive tight bounds on the contract value and present trading strategies exploiting the mispricing whenever the value lies outside the bounds. We estimate numerically the bounds of the reinsurance contract with real data from hurricane landings in Florida.

This Version: June 2014

Keywords: catastrophe events; jump processes; insurance products; derivative assets; stochastic dominance; non-convex payoff

JEL Classification: G13
1 Introduction

Catastrophe (CAT) derivatives are financial instruments indexed on a rare events process, a physical event whose occurrence reduces aggregate wealth and/or consumption (a catastrophe event). Such instruments have appeared often in recent years, fulfilling through securitization the financing needs of the insurance industry.\(^1\) These CAT products typically pay a cash flow to their holders that is conditional on the catastrophe event occurring and, whose size is, in most cases, proportional to the intensity of the event or to the financial losses incurred by the holder as a result of the catastrophe event. Such financial instruments may trade over the counter or in organized exchanges. They include catastrophe bonds, whose coupon and/or principal are reduced by pre-specified amounts due to the occurrence of the CAT event, futures contracts whose payments are proportional to the difference of the events intensity from a reference value, and reinsurance contracts that most often include a deductible and a ceiling on payments as a result of the CAT event.

In this paper we model the pricing of the catastrophe derivative as a contingent claim on the underlying accumulated loss due to hurricane landings in a geographical region. We apply a recently introduced valuation methodology for such contingent claims that recognizes the fundamental incompleteness of financial markets arising from the occurrence of rare events. Unlike most studies on such derivatives this method does not assume that the CAT event risk can be diversified away, as in the classic Merton (1976) study. We argue that such an assumption is fundamentally contradicted by empirical evidence showing values of CAT-indexed financial instruments far in excess of their expected payoffs.\(^2\) We apply the method to the valuation of a contingent claim with a non-convex payoff, for which closed form expressions are not available. We also argue that the continuous time approach that dominates most of the derivatives literature is not suitable for CAT events dependent on physical disasters, such as the hurricane contracts that we consider. For this reason we adopt a discrete time framework and derive tight bounds on the value of a reinsurance contract on CAT event losses that depend solely on the observed price of a futures contract indexed on the intensity of the event.

For our valuation we use the theoretical framework of the stochastic dominance (SD) methodology that was introduced to the study of CAT derivatives by Perrakis and Boloorforoosh (2013, PB), itself based on earlier literature.\(^3\) That literature was limited to the valuation of contingent claims with convex payoffs, for which closed form expressions are available. For this reason, the reinsurance contract in that earlier study could be valued only if the contract had a deductible or a cap, but not both. Further, the derived bounds on the contract only used information from the distribution of the CAT event amplitude and were, for that reason, relatively wide. Here, by contrast, we value the more realistic case of a contract with both a deductible and a cap, and we derive tight bounds by also using data from the futures market on CAT events. Last, we adopt a recursive discrete time approach, which recognizes

---

\(^1\)See MMC Securities (2005).

\(^2\)See the World Bank study by Lane and Mahul (2008).

that the CAT event of a hurricane landing develops over a number of days and, therefore, the number of such events in a given time frame is perforce limited. Our method is independent of distributional assumptions on the CAT event amplitude and can be generalized without reformulation to a Markovian process that may include dependence between the amplitude distribution and the frequency of occurrence of the event.

Although the underlying process is not a traded financial instrument, there exist futures contracts that allow the trading of the intensity of the landed hurricanes. Such contracts were introduced by the Chicago Board of Trade as early as 1992. They trade both in the organized exchanges, but also in a very active over-the-counter market in such instruments. We demonstrate that we can derive tight bounds on the price of the CAT derivative from observed futures prices and the physical distribution of the CAT event amplitude. In the numerical calculation we find reasonably tight bounds for the price of the reinsurance contract conditional on the assumed price of the futures contract, which can be represented by a linear premium on the expected amplitude of the CAT event. The fact that this single parameter is sufficient to accurately price the contract illustrates the major advantage of the SD method vis–vis alternative approaches that assume the knowledge of the entire risk adjusted distribution of the CAT event, allegedly extracted from other priced CAT instruments. Aside from the fact that such an extraction assumes away the formidable data problems and accepts the efficiency of a non-transparent market of over-the-counter instruments, our numerical results show that the admissible values of the CAT reinsurance contract are strongly dependent on the characteristics of the contract and cannot be represented by an event-dependent markup over the Merton value, which often lies far outside the derived bounds.

Apart from the derivation of the reinsurance contract values on the basis of the futures contract price there are several other important contributions of this paper over and above the PB study. First, we introduce the discrete time valuation methodology for CAT instruments with non-convex payoffs and show that such instruments values cannot be replicated with options with convex payoffs, as in the arbitrage-based approaches. In turn, this methodology has applications to other commonly encountered instruments such as CAT-indexed bonds whose payoffs resemble a combination of a bond and a digital option. We also show that in many cases no closed form expression arises for continuous time valuation quite apart from the limitations of the physical CAT process. Last but not least, we show that the same bounds can be derived from arbitrage strategies using second-degree stochastic dominance considerations. These strategies can exploit the mispricing of the CAT instrument whenever it lies outside the bounds.

Catastrophe financial instruments have attracted relatively little interest in the mainstream financial literature, and most contributions have appeared in the insurance literature. A

---

4Hurricane futures and options contracts are trading in the Chicago Mercantile Index (CME). These contracts are indexed on the CHI, the CME hurricane index, but their prices are available only to subscribers or data purchasers. The CME indicated in a private communication that most such instruments are traded via blocks as option structures, and that brokers post markets in these niche products. For these reasons we did not use actual futures price data, as we discuss further on in Section 4 of this paper.

5See, for instance, the discussion on market incompleteness in Geman and Yor (1997, p. 187) and in Bakshi and Madan (2002, pp. 107-108).

6See, for instance, Dassios and Jang (2002), Jaimungal and Wang (2006), Lee and Yu (2007), and Lin and
catastrophe event is by definition a rare event and as such is modeled mathematically as a
pure jump process, with Poisson arrivals and generally distributed amplitudes.\textsuperscript{7} The losses
inflicted by the event are generally considered proportional to the amplitude of a measurable
characteristic of the event, such as the intensity of a hurricane or the extent of a flood. Thus,
the probability distribution of the losses conditional on the occurrence of the event, as well
as the frequency of the event, can be extracted reasonably accurately from past data.

More contentious are the valuations of these losses, on which earlier studies have followed
two divergent paths. Although it is well-known at least since Merton (1976) that rare
events introduce incompleteness into the financial markets, this incompleteness has often
been ignored or assumed away in mathematical insurance studies that rely on continuous time
models paying scant attention to the underlying economic reasoning. In most of these models
it is assumed that the CAT event risk is fully diversifiable through an efficient reinsurance
market and a unique arbitrage-based price for the CAT financial instrument, equal to the
expectation of the losses with the properly estimated financial process. Nonetheless, this
assumption is clearly not applicable if catastrophe risk has economy-wide implications.\textsuperscript{8} More
to the point, the diversifiable CAT risk assumption is flatly contradicted by the empirical
evidence presented by Lane and Mahul in a 2008 World Bank survey of the markets for CAT
instruments over the period 1997-2008, in which the average instrument traded at 2.69 times
the expected loss! Clearly, with such multipliers any accuracy in modelling the probabilistic
structure of the event is dwarfed by the error in the economic assumption.

On the other hand, the more traditional insurance literature does recognize the existence
of an insurance premium on the expected loss, but relies mostly on stylized single-period
models that are not useful in pricing financial instruments on the basis of their cash flows.\textsuperscript{9} In-between the two literature strains are some studies that pay lip service to the economic
valuation of the cash flows, mostly by following the option pricing literature under jump
diffusion asset dynamics as in Bates (1991).\textsuperscript{10} Unfortunately these valuation exercises rely on
market equilibrium arguments using particular utility functions to transform the probability
distributions and make the values dependent on the risk aversion of a representative investor.
Since there is no general agreement on the size of this parameter,\textsuperscript{11} the transformation is left
unspecified in most empirical applications of jump-diffusion processes.

The incomplete markets bounding approach introduced by PB is an alternative to the equi-
librium results that does not require fundamentally unobservable elements. It is based on the
only assumption of the monotonicity of the pricing kernel used in valuing financial instru-
ments with respect to the CAT event amplitude. It is, therefore, well-suited to the valuation
of contingent claims indexed on a catastrophe event, given the market incompleteness that
it gives rise to, as well as the fact that the contingent claims are negative beta assets, whose

\textsuperscript{7}See Geman and Yor (1997), Froot (2001), and Muermann (2003).
\textsuperscript{8}This was already known from the option pricing literature in the presence of event risk. See, for instance,
Bates (1991). Further, Ibragimov et al. (2009) note that the efficient reinsurance assumption may not be
satisfied in real markets even if catastrophe risk does not have economy-wide impact.
\textsuperscript{9}See, for instance, Barrieu and Loubergé (2009) and Bernard and Tian (2009).
\textsuperscript{10}See Christensen and Schmidli (2000), Duan and Yu (2005) and Chang et al. (2010).
\textsuperscript{11}See Kocherlakota (1996), as well as the comments by Chang et al. (2010, p. 28, note 6).
cash flows increase when aggregate wealth decreases. Here we extend the PB results in a discrete time context, by showing that their SD approach can also be applied to claims whose payoff is not necessarily convex, and by tying it explicitly to the price of a CAT event futures contract that yields tight bounds on the value of the claim solely as functions of the futures price. The discrete time representation has the added advantage of modeling reality much more accurately than its continuous time counterpart: hurricanes and floods typically take several days to develop and land in a particular region, which puts stringent limits on the number of possible landings in any finite time interval.

In the next section we formulate the basic equations of the market model used in pricing the catastrophe instruments with a non-convex payoff in a single-period context and show that they cannot be derived from conventional call options. Section III derives an algorithm that extends the valuation results to any number of periods. Section IV discusses the estimation of the parameters and the numerical calculations of the bounds and compares the discrete to the continuous time model and to available empirical results. Section V concludes.

2 The Single-Period Model

It is well known since Merton (1976) that the market for rare event risk is incomplete, and the arbitrage valuation of a derivative instrument cannot yield a unique price. To value CAT derivatives we use the market equilibrium under the stochastic dominance assumptions formulated in previous studies: there exists at least one utility-maximizing risk averse investor (the trader) in the economy who holds a portfolio containing an index and the riskless asset, this particular investor is marginal in the derivative market, and the riskless rate is non-random. In this particular case it is also assumed that the index return depends linearly on the intensity of the CAT event.

The market equilibrium conditions for a trader holding a portfolio of the riskless asset and the index and maximizing recursively the expected utility of final wealth over a number of periods longer than the derivatives maturity were derived in an earlier study and will be only briefly summarized here. Let \( x_t + y_t \) denote the current value of the traders portfolio, with \( x_t \) and \( y_t \) denoting, respectively, the amounts invested in the riskless asset and the index. Let also \( R > 1 \) be the return of the riskless asset per period. Time is discrete \( t = 0,1,...,T \), with intervals of length \( \Delta t \). In each interval, the return of the index has the following form:

\[
\frac{y_{t+\Delta t} - y_t}{y_t} \equiv z_{t+\Delta t} = v_{t+\Delta t} + \gamma H \Delta N
\]

We assume that \( E_t[z_{t+\Delta t}] \geq R - 1 \). The term \( H \) represents the level of hurricane intensity, measured in CME Hurricane Index (CHI) units, and \( N \) is a Poisson counting process with

---


\(^{13}\)This condition will be defined more precisely in Section 4.

\(^{14}\)See Perrakis and Boloorforoosh (2013).
intensity with intensity $\lambda$. We consider a single-period model with horizon $\Delta t$, in which the probability of a hurricane landing in the area covered by the CAT derivative is $\lambda \Delta t$.\(^{15}\) Hence, the return $z_{t+\Delta t}$ is the convolution of the process $\gamma H_{t+\Delta t}$ with probability $\lambda \Delta t$ and $H_{t+\Delta t} = 0$ with probability $1 - \lambda \Delta t$, and of $\nu_{t+\Delta t}$, which is the index return component that is independent of the CAT event and whose distribution may depend on the current state level. The parameter $\gamma$ denotes the impact of the hurricane on the traders portfolio return. If $\gamma = 0$ then the CAT event risk is diversifiable and does not affect the traders optimal invested wealth, and we have a case similar to that of Merton (1976). In such a case the CAT event risk is not priced in equilibrium and we can obtain a unique price for the contingent claim, equal to the discounted expected payoff under the physical distribution of the underlying loss process. In our analysis we assume that $\gamma$ is negative, meaning that the CAT event risk is not diversifiable.\(^{16}\) Let also $F$ denote the futures price of a contract that matures at the end of the single-period horizon and whose payments are proportional to the hurricane intensity index. The contract execution is triggered by a hurricane landing, after which the contract expires and a new contract is issued till the end of the hurricane season. Without loss of generality we define $H_0$ as a hurricane intensity level of 0, the absence of a hurricane landing whose arrival triggers the futures contract maturity; alternatively, it is a below-hurricane-level wind intensity. With this definition we let $(q_i, H_i)$ be the combined distribution of wind intensity level and hurricane landing, with $q_0 = 1 - \lambda \Delta t$, $q_i = p_i \lambda \Delta t$, $i = 1, \ldots, n$.

Let $V_t \equiv \kappa \sum_{t=0}^{N_t} H_t$ denote the accumulated losses from the CAT event, assumed proportional to the hurricane intensity, where $\kappa$ represents the dollar loss per CHI units. As noted in the introduction, the accumulated loss process is a discrete time process, since the formation and landing of a hurricane takes time. If the trader also takes a marginal position in the reinsurance contract, $C_t(V_i)$, that expires at time $T \leq T'$, then the following relations characterize the market equilibrium in any single trading period $(t, t + \Delta t)$, assuming no market frictions:

$$
\begin{align*}
E_t[X(z_{t+\Delta t})] &= 1 \\
E_t[(1 + z_{t+\Delta t})X(z_{t+\Delta t})] &= R \\
C_t(V_t) &= \frac{1}{R} E_t[C_{t+\Delta t}(V_t + \kappa H_{t+\Delta t})X(z_{t+\Delta t})]
\end{align*}
$$

In (2.2) $X(z_{t+\Delta t})$ represents the pricing kernel or normalized rate of substitution of the trader evaluated at her optimal portfolio choice. Since the trader is assumed to be risk averse and since she has a marginal position in the contingent claim, it can be easily seen that the pricing kernel would be monotone non-increasing in $z_{t+\Delta t}$. Let also $\hat{X}_{t+\Delta t} = E_t[X_{t+\Delta t}, H_{t+\Delta t}]$ denote the pricing kernel conditional on the intensity of the landed hurricane. Since the hurricane is obviously an event that negatively affects aggregate consumption, as modeled by $\gamma < 0$ in (2.1), $\hat{X}_{t+\Delta t}$ is now non-decreasing in the intensity of the CAT event, with values $\hat{X}_i, i = 1, \ldots, n$, such that $\hat{X}_0 \leq \hat{X}_1 \leq \cdots \leq \hat{X}_n$.

\(^{15}\)Alternatively, we can set this probability equal to $1 - e^{-\lambda \Delta t} = \lambda \Delta t + o(\Delta t)$, in which case the probability of no landing is $e^{-\lambda \Delta t} = 1 - \lambda \Delta t + o(\Delta t)$.

\(^{16}\)We make no assumption regarding the value of $\gamma$ since its effect on the bounds is incorporated into the futures price.
First we consider a single-period case where there is one period left to the end of the hurricane season, and the reinsurance contract that expires in the next period is valid for only one hurricane landing.\textsuperscript{17} We model the reinsurance contract $C$, which is a contract against wind damages, in the form of a spread. We assume that the reinsurance contract has a deductible and a ceiling, corresponding to the hurricane intensities $H_i$ and $H_h$. The reinsurance contract has the following payoff, $C_T$, at the end of the hurricane season:

$$
C_{T_i} = 0, H_i \leq H_l \\
C_{T_i} = \kappa(H_i - H_l), H_l < H_i \leq H_h \\
C_{T_i} = \kappa(H_h - H_l), H_i > H_h
$$

(2.3)

The market equilibrium equations are, therefore, the following:

$$
\sum_{i=0}^{n} q_i \hat{X}_i = 1, \\
\sum_{i=0}^{n} q_i \hat{X}_i H_i = F, \\
\hat{X}_0 \leq \hat{X}_1 \leq \ldots \leq \hat{X}_n
$$

(2.4)

The last equation in (2.4) reflects the fact that any cash flows that accrue because of the hurricane event must be considered as an equivalent to a “negative beta” stock. Moreover, the price of this claim at a time of prior to the expiration of the contract is given by the following

$$
C = R^{-1} \sum_{i=0}^{n} q_i \hat{X}_i C_{T_i} = \kappa(\sum_{l=1}^{h-1} q_i \hat{X}_i (H_i - H_l)) + (H_h - H_l) \sum_{k=0}^{n} q_i \hat{X}_i R^{-1}
$$

(2.5)

Since the number of states, $n$, is obviously greater than 2, the market is incomplete, the pricing kernel is not unique, and no unique price can be defined by arbitrage methods alone. Following the linear programming (LP) approach pioneered by Ritchken (1985), we develop the tightest upper and lower bounds that the market equilibrium described in (2.4) can support. Further, the payoff is not convex\textsuperscript{18} with respect to the underlying random variable and the underlying asset is a negative beta security, implying that the expressions need to be modified. Nonetheless, the Ritchken approach can be easily adapted to account for the negative beta. In an appendix available from the authors on request it is shown that the bounds on the contingent claim are found by the following transformation of the market equilibrium. For a set of non-negative numbers $\epsilon_0, \ldots, \epsilon_n$, we set $\hat{X}_0 = \epsilon_0, \hat{X}_1 = \epsilon_0 + \epsilon_1, \ldots, \hat{X}_n = \sum_{i=0}^{n} \epsilon_i$, and we define $\hat{X}_i = \epsilon_i \sum_{k=i}^{n} q_k$. We also define the following conditional moments:

\textsuperscript{17}Alternatively, we may consider the single-period case as referring to the aggregate landings over a given time period, the convolution of individual hurricane landed intensities covered by a single futures contract. The more common case, in which the reinsurance contract covers the total loss during a hurricane season but each futures contract covers a single landing, will be examined in the multiperiod model.

\textsuperscript{18}The bounds are given by closed form expressions when the claims payoff is convex. See Perrakis and Boloorforoosh (2013, p. 3160).
\[
\bar{H}_i = \frac{\sum_{j=i}^{n} q_j H_j}{\sum_j q_j}, \quad \bar{C}_{Ti} = \frac{\sum_{j=i}^{n} q_j C_{Tj}}{\sum_j q_j}, i = 0, \ldots, n. \tag{2.6}
\]

Replacing these relations into (2.4), we have the following transformed market equilibrium conditions:

\[
\begin{align*}
\sum_0^n \tilde{X}_i &= 1, \\
\sum_0^n \tilde{X}_i \bar{H}_i &= F, \\
\tilde{X}_i &\geq 0, i = 0, \ldots, n. \tag{2.7}
\end{align*}
\]

Clearly \( \bar{H}_0 = E[H_i], \bar{H}_n = H_n \), and similarly \( \bar{C}_{T0} = E[C_T], \bar{C}_{Ti} = \kappa(H_h - H_l), i \geq h \). The price of the contingent claim represented by the reinsurance contract in (2.5) becomes now equal to

\[
C = R^{-1} \sum_0^n \tilde{X}_i \bar{C}_{Ti}, \tag{2.8}
\]

for a set \( \{\tilde{X}_i\} \) satisfying relations (2.7), which represents an admissible martingale probability. We want to determine the option bounds that this equilibrium supports, which are given as the solution of the following programs:

\[
C_{\text{max}} = R^{-1} \max_{\tilde{X}_i} \sum_0^n \tilde{X}_i \bar{C}_{Ti}, \quad C_{\text{min}} = R^{-1} \min_{\tilde{X}_i} \sum_0^n \tilde{X}_i \bar{C}_{Ti} \tag{2.9}
\]

subject to (2.7).

If the contract has no deductible or no ceiling then the payoff is respectively concave or convex with respect to the hurricane intensity. In such cases the solution to (2.9) can be found by an application of the dual of the LP (2.9), as in Ritchken (1985), which relies on such payoff shapes. Here it is possible to extend the the LP approach by means of the following result.

**Lemma 1.** The graph of the expected conditional payoff, \( \bar{C}_{Ti} \), as a function of the expected conditional intensity, \( \bar{H}_i \), is concave over the region \( \bar{H}_i > \bar{H}_l \), while it is convex for \( \bar{H}_i \leq \bar{H}_l \).

**Proof.** See Appendix A. \qed

Given this result we may now find the bounds on the admissible values of the reinsurance contract as functions of the futures price \( F \). This price plays the role of the stock price in conventional financial derivatives, and it turns out that the effects of market incompleteness can be represented by the excess of \( F \) over \( \bar{H}_0 \), the expected hurricane intensity. The bounds of the reinsurance contract are given by the following result.
Proposition 1. The upper and lower bounds \( C_{\text{max}}(F) \) and \( C_{\text{min}}(F) \) of the reinsurance contract, the solutions of the LP (2.6)-(2.9), depend on the size of the futures price \( F \) relative to the deductible and the ceiling of the contract. The bounds are found as the intersection of the vertical line stemming from \( F \) and the boundaries of the convex hull of the conditional payoff, as illustrated in Figure 2.1 and are described in equations (B.1)-(B.3) in the appendix.

Proof. See Appendix B.

\[ \kappa(H_h - H_l) \]

Figure 2.1: Convex Hull of the Conditional Payoff

The results of Proposition 1 apply to the valuation of a reinsurance contract on the accumulated hurricane losses, whose payoff has the shape of a vertical spread. This payoff, however, can be replicated with a long position in a call option with strike price equal to the deductible of the reinsurance contract, and a short position in a similar call option with strike price equal to the ceiling of the reinsurance contract. In complete markets, and in valuation methods that rely on a representative investor, this replication is sufficient to price the vertical spread. In the stochastic dominance approach the replicating call options upper and lower bounds can be calculated using the method presented by Perrakis and Boloorforoosh (2013), as the discounted expected option payoff under the upper and lower bound distributions which are available in closed form. The bounds on the reinsurance contract can thus be obtained from the bounds on the two calls. In Appendix C we show, both theoretically and numerically that except for the trivial cases where the deductible is very low and/or the ceiling is very high, the method presented in this paper yields considerably tighter bounds.

Although the bounds were derived under the assumption that the amplitude distribution is discrete, the derived expressions can be obviously adapted without reformulation to a distribution with compact support. Further, the valuation method presented here can be easily extended with very little reformulation to all other rare event instruments with non-convex payoff. These include the important case of catastrophe bonds issued by an insurer. The latter are straight bonds for which the issuer retains a digital call option in the form of a fixed reduction from the principal in case of a hurricane landing.
3 Multiperiod Analysis

In a multiperiod analysis the market equilibrium equations (2.4) or (2.7) remain the same. What changes is the nature of the payoffs to the reinsurance contract, which are similar to those of an Asian option. The reinsurance contract typically applies to the cumulative losses arising during the entire hurricane season. Hence, at any time during the season the state of the contract is described by the accumulated losses up to that point, whose amount may change during any time period due to a new hurricane landing. As for the CME hurricane futures, we assume that after each hurricane landing the existing futures contract matures, and a new futures contract is initiated. We assume that the reinsurance contract is of the European type and is exercisable only at $T$.

Let $t = 0, 1, \ldots, T$ denote the sequence of time points from the beginning to the end of the hurricane season. As defined in the previous section, $V_t$ denotes the accumulate losses till time $t$ and $C_t(V_t|F_t)$ is the corresponding value of the contract at $t$ if the observed futures price is $F_t$. We assume that the hurricane landings arrive independently with probabilities $\lambda \Delta t$ per period and their intensities are independent and identically distributed (iid) random variables with distributions $(p_i, H_i)$, $i = 0, 1, \ldots, n$. Further, we assume that in any period there cannot be more than one hurricane landing. Letting again $q_0 = 1 - \lambda \Delta t$, $q_i = p_i \lambda \Delta t$, $i = 1, \ldots, n$ denote the combined probabilities of a hurricane landing and of the hurricane intensity we have

$$V_0 = 0, V_t = V_{t-1} + \kappa H_t, H_t \sim (q_i, H_i), i = 1, \ldots, n, t = 1, \ldots, T$$

(3.1)

Observe that barring a shift in the pricing kernel due to a shift in preferences the iid assumption on hurricane occurrence and intensity implies from (2.4) that $F_t$ will remain the same for all $t$. Nonetheless, our formulation is sufficiently flexible to accommodate predictably seasonal shifts in the hurricane intensity distribution within the reinsurance contracts maturity, which will also be reflected by predictable changes in the futures price. It is also possible to extend the model by relaxing the iid assumption of hurricane intensities and replace it by a Markovian one, in which the intensity of a hurricane depends on that of the previous one.

At time $t = T - 1$ the bounds on the value of the reinsurance contract can be found by applying the analysis of the previous section, with the important caveat that the terminal payoffs depend on the accumulated losses $V_{T-1}$ which is the state variable. Both the deductible and the ceiling on cash flows are imposed on the terminal accumulated losses $V_T$, at two levels $\overline{V}_l$ and $\overline{V}_h$. This simplifies the single-period analysis. Thus, for $V_{T-1} \geq \overline{V}_h$ we clearly have $C_T = \overline{V}_h - \overline{V}_l$ for all values of $V_T$, while for $V_{T-1} \geq \overline{V}_l$ the terminal payoff is concave (linear and then constant) in $H_t$ for all values of $V_T$. For $V_{T-1} < \overline{V}_l$ we redefine the values $H_l(V_{T-1}) \geq H_0$ and $H_h(V_{T-1}) < H_n$, with $l(V_{T-1})$ and $h(V_{T-1})$ being the smallest integers such that $\kappa H_l + V_{T-1} \geq \overline{V}_l$, $\kappa H_h + V_{T-1} \geq \overline{V}_h$, with the payoff having the same shape as in the single-period problem; note that both $l(V_{T-1})$ and $h(V_{T-1})$ are decreasing functions.

---

19This follows from the fact that the value of the futures contract is zero at any time point and the contract can be closed at any time point. Hence, the futures price is equal to the value of all possible cash flows that accrue to the contracts position.
With these redefinitions the payoff of the contingent claim $C_{T-1}(V_{T-1}|F_{T-1})$ becomes at time $T$ a function that has the same shape as in Figure 2.1, with the starting point displaced by the amount $V_{t-1}$:

$$V_{T-1} < \bar{V}_l :$$

$$C_T = 0, H_T \leq H_l; C_T = V_{T-1} + \kappa H_T - \bar{V}_l, H_T \in (H_l, H_h];$$

$$\bar{V}_l \leq V_{T-1} < \bar{V}_h :$$

$$C_T = V_{T-1} + \kappa H_T - \bar{V}_l, H_T \leq H_h; C_T = \bar{V}_h - \bar{V}_l, H_T > H_h;$$

$$V_{T-1} \geq \bar{V}_h :$$

$$C_T = \bar{V}_h - \bar{V}_l, \forall H_T$$  \hspace{1cm} (3.2)

Given this payoff, the following results allow us to estimate the upper and lower bounds $C_{T-1,max}(V_{T-1}|F_{T-1})$ and $C_{T-1,min}(V_{T-1}|F_{T-1})$ on the value $C_{T-1}(V_{T-1}|F_{T-1})$.

**Lemma 2.** The bounds $C_{T-1,max}(V_{T-1}|F_{T-1})$ and $C_{T-1,min}(V_{T-1}|F_{T-1})$ on the reinsurance contract lie on the convex hull of the conditional payoffs, and their values depend on $V_{T-1}$.

**Proof.** The equations representing the boundary of the convex hull and the proof are presented in Appendix D.

**Lemma 3.** For any given futures price $F_{T-1}$ the bounds on the contingent claim $C_{T-1}(V_{T-1}|F_{T-1})$ are increasing functions of $V_{T-1}$, concave for the upper and convex for the lower, and are identified by the intersection of the vertical line stemming from the futures price with the convex hull of the conditional payoff.

**Proof.** The equations representing the bounds and the proof are presented in Appendix E.

We may now formulate the recursive problem that derives the multiperiod bounds for the value of the contingent claim $C_t(V_t|F_t)$ corresponding to the reinsurance contract. Assuming no shift in preferences, so that the futures price remains the same, $F_t = F$ for all $t$, at any time $t < T - 1$ the capital market equilibrium must satisfy the following system:

$$\sum_{i=0}^{n} q_i \hat{X}_i = R^{-1}, \sum_{i=0}^{n} q_i \hat{X}_i H_i = FR^{-1},$$

$$C_t(V_t|F) = R^{-1} \sum_{i=0}^{n} q_i X_i C_{t+1}(V_{t+1}|F) = R^{-1} \sum_{i=0}^{n} q_i \hat{X}_i C_{t+1}(V_t + \kappa H_i|F)$$ \hspace{1cm} (3.3)

$$\hat{X}_0 \leq \hat{X}_1 \leq \ldots \leq \hat{X}_n$$

Define also the following values, the counterpart of (2.6):

$$\underline{C}_{t+1,\alpha,i} = \frac{\sum_{j=i}^{n} q_j C_{t+1,\alpha,j}(V_t + \kappa H_j|F)}{\sum_{j=i}^{n} q_j}, \alpha = \max, \min, i = 0, \ldots, n.$$  \hspace{1cm} (3.4)
We may then prove the following result.

**Proposition 2.** The value \( C_t(V_t|F) \) of the reinsurance contract lies between the following recursive bounds \( C_{t,\text{min}}(V_t|F) \) and \( C_{t,\text{max}}(V_t|F) \) for all \( t < T \):

\[
C_{t,\text{max}}(V_t|F) = \frac{H_i^* + 1 - F}{H_i^* + 1 - H_i} C_{t+1,\text{max},i^*} + \frac{F - H_i^*}{H_i^* + 1 - H_i} C_{t+1,\text{max},i^*+1},
\]

\[
C_{t,\text{min}}(V_t|F) = \frac{H_n - F}{H_n - H_0} C_{t+1,\text{min},i} + \frac{F - H_0}{H_n - H_0} C_{t+1,\text{min},n}, \quad V_{T-1} < V_t < V_T
\]

(3.5)

where \( C_{T-1,\text{max}}(V_{T-1}|F) \) and \( C_{T-1,\text{min}}(V_{T-1}|F) \) are as defined in Lemma 3, and \( i^* \) is a state such that \( H_i^* \leq F \leq H_i^* + 1 \).

**Proof.** See Appendix F. \( \square \)

The recursive evaluation of the reinsurance contracts bounds is computationally very simple, in spite of the complexity of the notation in representing the convex hull of the one-period payoff. This hull remains the same at every recursion and every state space node, and what changes is the starting point that is a function of the cumulative losses to that node, which determines the location of the futures price \( F \). The convexification of the bounds is also maintained in every recursion, thus allowing the closed form expression of the LP solution as derived by Ritchken (1985). Unlike the continuous time derivatives with convex payoffs examined in Perrakis and Boloorforoosh (2013), there is no general closed form solution for the bounds on the value of the contract, because the upper and lower bound distributions are now state dependent. Nevertheless, the evolution of the value is Markovian and can be easily estimated for realistic numbers of partitions in the time subdivision.\(^{20}\)

### 4 The Arbitrage Derivation of the CAT Bounds

In this section we prove that the derived bounds on the CAT reinsurance contract are also arbitrage bounds, in the sense that their violation indicates a profit opportunity for the trader. Such an opportunity differs from conventional arbitrage, insofar as the SD bounds violations are exploited by adding to the traders wealth zero net cost portfolios that contain the mispriced contract and that offer superior risk-adjusted returns to the trader independent from his/her wealth or attitude towards risk. The derivation of the bounds by this type of arbitrage is an alternative to the LP approach used in this paper and was presented in detail in Oancea and Perrakis (2014). Nonetheless, that derivation was applicable to positive beta securities and relied heavily on the convexity of the payoffs, resulting in closed form

\(^{20}\)For the same reason it is not easy to visualize the continuous time limit of the bounds for such derivatives with non-convex payoffs. Such continuous time limits, however, are not realistic for physical CAT events such as hurricanes, earthquakes or floods.
expressions for the arbitrage portfolios. For this reason we present here the single-period
derivation of the reinsurance contract bounds by arbitrage, since the zero net cost portfolios
necessary for the exploitation of contract mispricing need to be adapted to the specific
problem. We shall derive the arbitrage strategies for the case shown in Figure 2.1, with
other cases left as an exercise.

For expository purposes and without loss of generality we assume, unlike the LP approach,
that the landed hurricane intensity has a distribution \( P(H) \) with compact support \( H \in [H_1, H_n] \), \( H_1 \geq 0, H_n < \infty \). The CAT event has amplitude 0 with probability \( 1 - \lambda \Delta t \) and \( H \) with distribution \( \lambda \Delta t P(H) \). We also assume, without loss of generality, that the
universe of traders consists of \( \Pi \) identical agents, in which case the marketed contract is
\( C_{\Pi}(V_{T-1}) = \frac{1}{\Pi} C(V_{T-1}) \). Setting for simplicity \( V_{T-1} = 0 \) and defining the multipliers \( \kappa_{\Pi} = \frac{\kappa}{\Pi} \),
our valuation reduces to the estimation of the bounds of the contingent claim \( C_{\Pi} \). The payoff,
denoted by \( C_{\Pi T}(H) \), has the same functional form as in (2.3) with \( \kappa_{\Pi} \) replacing \( \kappa \) conditional
on the occurrence of a hurricane landing, and 0 otherwise. Let also \( \Omega(x_t + y_t) \) denote the traders value function, the maximized expected utility of her portfolio at time \( t = T-1 \), which
is increasing and concave. By definition, \( \Omega(x_{T-1} + y_{T-1}) = E_{T-1}[\Omega(x_{T-1}R + y_{T-1}(1 + z_T))] \),
where the portfolio \( (x'_{T-1}, y'_{T-1}) \) is the optimally selected asset allocation at \( T-1 \). For the
index return given by (2.1) we also define the function \( W_{T-1}(H) \) as follows:

\[
\Omega(x_{T-1} + y_{T-1}) = E_{T-1}[E[\Omega(x'_{T-1}R + y'_{T-1}(1 + z_T))|v_T]] = E_{T-1}[W_{T-1}(H)] \tag{4.1}
\]

From the properties of the value function and the index return (2.1) it follows that the
derivative of the function \( W_{T-1}(H) \) is increasing in the hurricane intensity, a property that
will be important in deriving the bounds.

To derive the upper bound we assume that the trader can short the contract for a price of
\( C \), which is invested in the cash account. Suppose the trader shorts the contract and also
adopts a long position in \( \frac{CR}{MF} \) futures contracts, whose price is \( F \) and whose multiplier is
denoted by \( M \). This is a zero net cost position, corresponding to the short contract together
with a long position in an instrument with payoff proportional to landed hurricane intensity,
with coefficient \( \frac{CR}{F} \), which is strictly less than \( \kappa_N \) to avoid arbitrage between the futures
market and the reinsurance contract; this trader is termed the C-trader and the zero net
cost portfolio is termed \( h(H) \). For a properly priced contract such a C-trader should not
be able to increase her value function over that of an unspecified risk averse trader with
identical characteristics and wealth positions. Let \( \Omega^C(x_t + y_t) \) denote the value function of
the C-trader and set

\[
\Delta_{T-1} = \Omega^C(x_{T-1} + y_{T-1}) - \Omega(x_{T-1} + y_{T-1}) \tag{4.2}
\]

This difference will certainly not increase if the C-trader adopts the same portfolio revision
policy as the regular trader. We then have:
\[ \Delta_{T-1} = \Omega^C(x_{T-1} + y_{T-1}) - \Omega(x_{T-1} + y_{T-1}) = E_{T-1}[W_{T-1}^C(H)] - E_{T-1}[W_{T-1}(H)] \geq E_{T-1}[W_{T-1}(H + h(H)) - W_{T-1}(H)] \geq E_{T-1}[W_{T-1}^1h(H)] \]  

(4.3)

In (4.3) the term \( W_{T-1}^1 \) denotes the derivative of \( W_{T-1}(H + h(H)) \). Since the trader is by assumption marginal in the derivative market, \( W_{T-1}^1 \) is also increasing in the hurricane intensity \( H \). Since \( h(H) \geq 0 \) by construction when there is no hurricane landing, we concentrate on the shape of \( h(H) \) conditional on a landing. Note that \( h(H) \) is always initially increasing for \( H \in [H_1, H_1] \), decreasing for \( H_1 < h \leq H_h \), and increasing again for \( H \in [H_h, H_n] \). Its zeroes are at most two beyond the origin and are parameter-dependent, but for the case shown in Figure 2.1 there are exactly two zeroes, denoted by \( H^* \) and \( \tilde{H} \). We then have

\[ \Delta_{T-1} \geq E_{T-1}[W_{T-1}^1h(H)] \geq \lambda \Delta t \int_{H^*}^{H_n} W_{T-1}^1h(H)dP(H) \]  

(4.4)

So, to assess whether \( \Delta_{T-1} \geq 0 \) it suffices to consider only the last term of (4.4). We have:

\[ \int_{H^*}^{H_n} W_{T-1}^1h(H)dP(H) \geq \int_{H^*}^{H_n} \int_{H^*}^{H_n} h(H)dP(H) \]  

(4.5)

Substituting \( h(H) = \frac{CRH}{F} - C_{NT}(H) \) into the last integral of (4.5) and dividing by \( P(H^*) \) we conclude that \( \Delta_{T-1} \geq 0 \) unless \( C \leq \frac{1}{\bar{R}} \frac{FE[C_{NT}(H) \geq H^*]}{EF[H \geq H^*]} = \frac{E[C_{NT}(H) \geq H^*]}{E[H \geq H^*]} \). This last expression is, however, identical to the upper bound derived in Proposition 1 and shown in the second expression of (B.1) and (B.3).

The derivation of the lower bound is also straightforward, albeit a little more complex. The arbitrage strategy here is to purchase the reinsurance contract from the cash account and adopt a short position in \( \frac{CR}{MF}(1 + \alpha) \), \( \alpha > 0 \) futures contracts, with \( \frac{CR}{F}(1 + \alpha) < \kappa_N \). For the no landing case this strategy clearly dominates the no action case, so we concentrate on a landed hurricane with intensity \( H \). The payoff function \( h(H) \) is initially decreasing and equal to \( \frac{CR}{F} - \frac{CR}{F}(1 + \alpha)H \), becomes increasing at \( H \) and decreasing again at \( H_h \). Hence it can have at most three zeroes, and we choose \( \alpha \) by imposing the condition that the third zero should be at \( H_n \), so that

\[ \frac{CR}{F} - \frac{CR}{F}(1 + \alpha)H_n = C_{NT}(H_n) \]  

(4.6)

Letting \( H^* \) and \( \tilde{H} \) denote again the first and second zeroes, we repeat the reasoning leading to the derivation of the upper bound, which was given by the no stochastic dominance condition that implies \( E[h(H) \mid H \geq H^*] = 0 \). This same condition, together with (4.6), allows the
derivation of the lower bound, which is left as an exercise, together with the demonstration that it is the same expression given in (B.1) and (B.3).

The derivation procedure presented in this section also illustrates the trading strategies that can be used to exploit the mispricing in the reinsurance contract. When the price of the reinsurance contract violates the bounds derived in section III, the trader can increase her expected utility by adding a zero net cost position in the reinsurance contract and the futures contract to her portfolio, using the strategies described above.

In the next section we apply the bounds estimation procedure to a notional reinsurance contract on hurricane wind damage in the state of Florida. Without loss of generality it is assumed that the contract covers \textit{statewide} assessed damage, implying that the underlying random process is the landed intensity in the entire state.

\section{Estimation and Numerical Results}

The basic traded futures contract that will be used to value the hurricane reinsurance contract is the hurricane futures contract traded by the CME. It is indexed on the CHI, initially denoting the Carvill Hurricane Index, but the index was subsequently purchased by the CME and renamed CME Hurricane Index. Several types of contracts indexed on the CHI trade in the CME. The contract that is the most relevant for our purposes is the hurricane seasonal contract, quoted in CHI index points and representing the accumulated CHI total for all hurricanes that occur in a specified location within a given calendar year. The contract multiplier is $1000$ per CHI point.

We start by the following discretization of the time and state space. We divide the time $T - t$ until the maturity of the reinsurance into $N$ subdivisions of equal length $\Delta t$. We then construct a multinomial lattice with $n + 1$ branches emanating from each node to represent the cumulative catastrophe loss associated with the hurricane landings. Figure 5.1 depicts the cumulative loss process for two periods.

Hence, at any time $t$ where the cumulated loss is $V_t$ we have $n + 1$ possible outcomes for the next period cumulative loss at time $t + \Delta t$. The cumulative loss could stay the same, corresponding to the event of no hurricane landing, $(H_0 = 0)$, or it could go up to $V_t + \kappa H_i$, $i = 1, \ldots, n$. After $N$ periods we will have the probability distribution of the cumulative losses at time $T$.

We calibrate the underlying loss process based on the data available from CME for hurricane landings in the state of Florida for the period 1998-2007. We extract the conditional distribution of intensities, $(p_i, H_i)$, $i = 0, 1, \ldots, n$, from the histogram of CHI values for Florida landings for $n = 3$.\footnote{Since the lattice is non-recombining, after $N$ periods we have $(n + 1)^N$ states. This makes the computations very intensive and even infeasible for large values of $N$. Nevertheless, since it is not reasonable to expect large number of hurricane landings in a geographical region in every hurricane season, there is no need to consider large $N$.} Hence, the combined probability distribution of landing and intensities,
\((q_i, H_i)\), results in a quadriomial lattice for the underlying cumulative loss process. Moreover, we assume that in every period there can be only one hurricane landing.\(^{22}\) So, when the number of time subdivisions is \(N\), the minimum and maximum number of hurricane landings are zero and \(N\), respectively.

Conditional on landing, the hurricane intensity could be 3.42, 7.45, or 11.48 with equal probabilities, yielding an expected hurricane intensity conditional on landing equal to 7.45. The intensity of the Poisson process, \(\lambda\), representing the rate of hurricane landing arrival is estimated as the average number of landings per hurricane season in Florida and is set equal to 0.9. It is assumed, without loss of generality, that \(\lambda\) is constant (no seasonality within the contract period). Moreover, we estimate the loss coefficient \(\kappa = 0.41\) (in billion dollars per CHI) from the reported losses associated with nationwide landings during the period of 1975 – 2005.\(^{23}\) The average annual losses associated with hurricane landings in the state of Florida are equal to $6.7 billion. These numbers imply that if there were 100 insurers in the region with similar policies, each would be exposed to $67 million of losses per annum.

In order to calculate the bounds as described in the previous sections we need the price of the seasonal hurricane futures. Unfortunately, these contracts trade in a non-transparent market and mostly in block trades, implying that the market prices are illiquid and only approximately reliable as reflecting the “true” value of the hurricane event. Therefore, in our calculations we assume that the price of the seasonal futures contract is linked to the expected intensity of the hurricane according to the following linear relation.

\[
F = g \cdot E[H], \quad g > 1
\]  

(5.1)

The expectation in (5.1) is taken with respect to the combined distribution of landing and intensities. Since the distribution of intensities conditional on landing is assumed to be iid, \(F\) is independent of \(t\), and only depends on the time to maturity of the contract, and thus on the combined distribution of \(H\) at the end of the period. We report the bounds for a range of value of \(g\), which cover the values observed in past financial instruments indexed on wind damages.\(^{24}\)

After the lattice is constructed and calibrated, as described in the previous sections, we calculate the bounds recursively starting at time \(T - 1\). Figure 5.2 shows the recursive process for the underlying lattice depicted in Figure 5.1. As shown in the lattice we start from the maturity of the contract, for which the conditional payoff is known to be \(c\). Then by constructing the convex hull of the conditional payoffs as discussed in the previous sections, we calculate the bounds at time \(T - 1\). These bounds are then used to construct the convex

\(^{22}\)This assumption is justified because the formation and landing of hurricanes takes time, and it is a reasonable to assume that, in general, there cannot be more than one hurricane landing over a period of 15-20 days.

\(^{23}\)The coefficient was estimated from a linear regression of the hurricane losses on the intensity of the landed hurricanes in US, based on the data provided by CME for the period of 1975 – 2005.

\(^{24}\)See Lane and Mahul (2008, Table 5). In fact the average coefficients for wind risk insurance range from 1.79 to 3.97, depending on time and place.
hull and obtain the bounds at time $T - 2$. This recursive estimation of the bounds is repeated until we get the bounds at $T - t$.

[Figure 5.2 about here]

Figure 5.3 and Table 5.1 show the multiperiod bounds, along with the Merton price, for a reinsurance contract with six months left to maturity and with a deductible and ceiling of 1 and 10 billion dollars, respectively. The riskless return is set to $R = 1.01$ and we choose the parameters values for our base case as $g = 2$, $\lambda = 0.9$, and $\kappa = 0.41$. The two bounds and the Merton price increase as we increase the number of periods. This is because as we increase $N$ we also increase the number of potential hurricane landings during the term of the reinsurance contract. Moreover, as the number of subdivisions increases, the bounds become tighter. Furthermore, we can observe that the Merton price always lies below the lower bound, and thus the assumption of a diversifiable CAT risk, results in serious underpricing of the reinsurance contract whenever $g > 1$. Last but not least, we note that the value of the parameter $g$, the excess premium on the average intensity of the CAT event reflected in the CAT futures price, is not sufficient to determine on its own the value of the reinsurance contract as a function of its Merton bound. For instance, for $N = 2$ the midpoint of the bounds is 3.53, about 9% higher than the value of 3.24, the Merton value multiplied by $g$, which understates even the lower bound of 3.47.

[Table 5.1 about here]

[Figure 5.3 about here]

The extent of underpricing of the contract as a function of for our base case parameters is shown in Figure 5.4 and Table 5.2. When $g = 1$, the futures price is proportional to expected hurricane intensity, corresponding to full diversifiability of the CAT event risk. In this case, the two bounds coincide with the Merton price, which is the discounted expected payoff under the physical distribution of the underlying process. As noted in the introduction, such an assumption is at variance with the observed facts, as well as with recent theoretical models that show the regional specialization of insurance firms that are the purchasers of the reinsurance contract. As we increase $g$, thus increasing the premium of the future price over the expected hurricane intensity, the two bounds increase and become wider, indicating the market incompleteness with respect to the CAT event risk, while the Merton price remains the same. Further, the relation of the reinsurance value as a function of $g$ is not linear: as Figure 5.4 and Table 5.2 show, for the base case $N = 8$ the bounds midpoint is initially approximately proportional to $g$, but for higher values of $g$, multiplying $g$ by the Merton value overstates significantly the value of the reinsurance contract. This result, as well as the previously noted understating of the reinsurance contract value when $N = 2$ and $g = 2$, validate our approach in valuing such contracts by recognizing market incompleteness and the physical attributes of the CAT event.

[Table 5.2 about here]

[Figure 5.4 about here]

---

25We assume that the length of the hurricane season in Florida is six months.
Figure 5.5 and Table 5.3 evaluate the bounds for different numbers of recursions. All the bounds are calculated for the same set of parameters as above, and for $N = 8$, implying that there can be up to 8 hurricane landings in the time left to the maturity of the reinsurance contract in all cases. We calculate the bounds when there are 1, 2, 4, and 8 recursions (trading opportunities) available during the life of the reinsurance contract, where these trading opportunities are spaced with equal length from each other. One recursion corresponds to the single-period bounds, with the remaining entries corresponding to the indicated number of recursions. It is clear that increasing the number of recursions results in progressively tighter bounds for a given number of potential hurricane landings.

The comparison of single-period and multiperiod bounds for a given number of potential landings is explored further in Table 5.4 and Figure 5.6. We calculate the single-period bounds with the same underlying loss process as for the multi-period model and for the indicated number of potential landings. We calculate the terminal accumulated intensities and the associated probabilities and aggregate the states by summing over the probabilities of the states with the same intensity. Hence, we end up with the terminal distribution of landing and intensities that is the equivalent to the multiperiod convolution of $(q_i, H_i)$. We denote this distribution $(q'_i, H'_i)$ and set the price of the seasonal futures contract to $F' = g \cdot E[H']$. It is clear that the multi-period bounds are much tighter than their single-period counterparts. This indicates as expected, that recursive estimation reduces the gap between the two bounds. This gap would converge to a minimum at the continuous time limit if such a convergence were not precluded by the physical restrictions of the hurricane development process. Even if we ignore these restrictions, the lack of closed form expressions for the bounds precludes the estimation of their limiting values.

6 Summary and Conclusions

In this paper we have presented an approach to the pricing of CAT derivatives with non-convex payoffs that is significantly different from the established methodology in earlier studies. Almost all these studies follow the Merton (1976) assumption that the rare event risk is fully diversifiable and should not be priced in equilibrium. Alternatively, they assume that a unique risk-adjusted distribution of the CAT event can be extracted from other traded financial instruments indexed on the event distribution. In such a case a unique CAT derivative price can be obtained in all cases and non-convexity is not an issue, since the unique price can be replicated by the prices of derivatives whose payoff replicates that of the valued derivative. Our approach relies on recent literature suggesting that economic agents trading in CAT instruments (for instance, insurance companies) specialize locally and in
special types of CAT risks, and is consistent with empirical evidence that rejects decisively the Merton assumption. Hence, the CAT event risk is in general not fully diversifiable. In such a case our approach recognizes the market incompleteness introduced by the non-diversifiable CAT event and relies on stochastic dominance arguments to develop bounds on the CAT event derivatives relying solely on traded futures contracts on the CAT event. We show that our bounds cannot be derived by replicating the derivative with plain vanilla call and put options, and we present an efficient discrete time algorithm that can handle several empirically meaningful CAT event derivatives. We also argue that the continuous time approach is not relevant to financial instruments indexed on CAT events arising from physical phenomena. We apply our method to the pricing of a catastrophe reinsurance contract on the underlying cumulative hurricane losses in the state of Florida. We assume that the reinsurance contract has a deductible and a ceiling, and so is in the form of a vertical spread. Our theoretical analysis predicts that the call option price bounds would lie above the Merton price, with the distance depending on the price of a hurricane futures contract such as the ones offered by the CME or trading over the counter. We use realistic parameter values and show that the reinsurance contract produces tight bounds for all admissible values of the parameters. We show that the Merton price lies far below our bounds for almost all realistic values of the hurricane futures contract parameter, expressed as a multiple of the expected intensity of the CAT event. We also show that the dependence of our bounds on this parameter is non-linear and varies with contract characteristics, thus illustrating the pitfalls of neglecting or minimizing the importance of the CAT event systematic risk. Last but not least, our method is applicable with minimal adaptation to other important CAT financial instruments such as bonds indexed to CAT events.

Appendix

A Proof of Lemma 1

For the first part, we note that for \( H_i > H_h \) the unconditional payoff \( C_{T_i} \) is concave over the set of points \( H_i \), while the points \( (H_i, C_{T_i}) \) for \( H_i \leq H_h \) play no role in the computation of the points \( (\overline{H}_i, \overline{C}_{T_i}) \). Hence, concavity is preserved, since the points of the latter points are linear combinations of the former ones. For the points \( (\overline{H}_i, \overline{C}_{T_i}) \) in the domain \( \overline{H}_i \leq \overline{H}_I \), we consider the slope \( \frac{\Delta \overline{C}_{T_i}}{\Delta \overline{H}_i} = \frac{\overline{C}_{T_{i-1}} - \overline{C}_{T_{i-1-1}}}{\overline{H}_i - \overline{H}_{i-1}} \) where the conditional payoffs and intensities are replaced from (2.6). Simplifying, we find that this quantity is a fraction with the same numerator but with a denominator that decreases as a function of \( i \) in the region \( \overline{H}_i \leq \overline{H}_I \), implying the convexity of the function. \( \square \)
B Proof of Proposition 1

The following relations characterize the lower and upper boundaries of the convex hull of the conditional contract payoff as presented in Figure 2.1.

\[
\begin{align*}
C_{i,\text{upper}} &= \frac{C_{T_i^*} - C_{T_0}}{\bar{H}_i - \bar{H}_0} \cdot (\bar{H}_i - \bar{H}_0) + C_{T_0}, \ i \leq i^* \\
C_{i,\text{upper}} &= \overline{C}_{T_i}, \ i \in (i, h) \\
C_{i,\text{upper}} &= \kappa(H_h - H_i), \ i \geq h \\
C_{i,\text{lower}} &= \overline{C}_{T_i}, \ i \leq \tilde{i} \\
C_{i,\text{lower}} &= \overline{C}_{T_i^*} + \frac{\kappa(H_h - H_i) - \overline{C}_{T_i}}{H_n - \overline{H}_i} \cdot (\bar{H}_i - \overline{H}_i), \ i > \tilde{i}
\end{align*}
\]

(B.1)

where, \(l\) and \(h\) represent states corresponding to the hurricane intensities \(H_l\) and \(H_h\), respectively. Moreover, \(i^*\) and \(\tilde{i}\) are the tangency points of the non-convex payoff on the lower and upper boundaries of the convex hull, respectively, and can be found as follows:

\[
\begin{align*}
\tilde{i} &= \arg \max_i \overline{C}_{T_i}, \ i \leq i^* \\
\tilde{i} &= \arg \max_i \left( \frac{\kappa(H_h - H_i) - \overline{C}_{T_i}}{H_n - \overline{H}_i} \right) \quad (B.2)
\end{align*}
\]

The bounds on the contingent claim can be found based on the value of \(F\) relative to the deductible and ceiling of the payoff, as follows:

For \(F \leq \overline{H}_h\) we have:

\[
\begin{align*}
C_{\text{max}} &= R^{-1} C_{i,F,\text{upper}}, \ C_{\text{min}} = R^{-1} \left[ \overline{C}_{T_i^*} + \frac{\kappa(H_h - H_i) - \overline{C}_{T_i^*}}{H_n - \overline{H}_i} (F - \overline{H}_i) \right] \\
C_{i,F,\text{upper}} &= \overline{H}_{i^*+1} - \frac{F - \overline{H}_{i^*}}{\overline{H}_{i^*+1} - \overline{H}_{i^*}} C_{i^*,\text{upper}} + \frac{F - \overline{H}_{i^*}}{\overline{H}_{i^*+1} - \overline{H}_{i^*}} C_{i^*,1,\text{upper}}, \quad (B.3)
\end{align*}
\]

For \(F \in (\overline{H}_h, \overline{H}_h)\) the lower bound remains unchanged while the upper bound becomes \(C_{\text{max}} = R^{-1} \kappa(H_h - H_i)\). Similarly, for \(F \in (\overline{H}_i, \overline{H}_i)\) the lower bound remains unchanged while the upper bound is still given by (B.3), but now \(C_{i,\text{upper}} = \frac{\overline{C}_{T_0} - \overline{C}_{T_0}}{\overline{H}_i - \overline{H}_0} \cdot (\overline{H}_i - \overline{H}_0) + \overline{C}_{T_0}\).

Last, for \(F \in (\overline{H}_0, \overline{H}_0)\) the upper bound is the same as for \(F \in (\overline{H}_i, \overline{H}_i)\), but for the lower bound we now have \(C_{\text{min}} = R^{-1} \overline{C}_{T_i,\text{lower}}\) and \(C_{i,F,\text{lower}} = \frac{\overline{H}_{i^*+1} - F}{\overline{H}_{i^*+1} - \overline{H}_{i^*}} \overline{C}_{T_i^*} + \frac{F - \overline{H}_{i^*}}{\overline{H}_{i^*+1} - \overline{H}_{i^*}} C_{T_i^*+1}\), where \(i^*\) is chosen such that \(\overline{H}_{i^*} \leq F \leq \overline{H}_{i^*+1}\).

Proof: By conditions (2.7)-(2.8) all admissible points \(C\) belong to the convex hull of the points \(\overline{C}_{T_i}\), the smallest convex space containing all such points. The boundaries of the convex hull is defined by equations (B.1) and (B.2), which are illustrated in Figure refconvexhull.

Similarly, all admissible sets \(\{\tilde{X}_i\}\) satisfying (2.7) lie within the convex hull and on the vertical line shown, which satisfies the constraint \(\sum_0^n \tilde{X}_i \overline{H}_i = F\). Hence, the upper and lower
bounds of $C$ are found from the intersection of the line with the upper and lower boundaries of the hull, $C_{i,\text{upper}}(H_i)$ and $C_{i,\text{lower}}(H_i)$ respectively. These intersections, however, yield (B.3), as well as the alternative cases that arise from the size of the futures price $F$ as described in the Proposition.

\[ \text{Tightness of the Bounds} \]

Let $C$ denote the reinsurance contract on the accumulated hurricane losses with a deductible and a ceiling equal to $H_l$ and $H_h$, respectively. The payoff to the reinsurance contract has the shape of a spread and can be replicated by two call options on the same underlying process. Consider a long position in a call option, $C_1$, with strike price equal to $H_l$, and a short position in a call option, $C_2$, with strike price $H_h$. The payoff to the combined position, $c_1 - c_2$, replicates the payoff to the reinsurance contract, $c$.

The upper bound of $C_1$ can be found as the solution to the following LP, subject to the market equilibrium conditions, as described in the single-period model.

\[
C_{1,\text{max}} = \max_{\{\tilde{X}_i\}} \sum q_i \tilde{X}_i c_{1,i} = \sum q_i X^*_i c_{1,i} \tag{C.1}
\]

Similarly, the lower bound of $C_2$ is the following:

\[
C_{1,\text{min}} = \min_{\{\tilde{X}_i\}} \sum q_i \tilde{X}_i c_{2,i} = \sum q_i X^*_i c_{2,i} \tag{C.2}
\]

where, $X^*_i$ and $X^*_2$ are the values of the pricing kernel corresponding to the solution of the maximization and minimization, respectively. Thus, the upper bound of the reinsurance contract, obtained from the replicating portfolio, would be equal to $C_{2,\text{max}} - C_{1,\text{min}}$.

On the other hand, the upper bound of the reinsurance contract can be obtained as the solution to the following LP, subject to the market equilibrium conditions:

\[
C_{\text{max}} = \max_{\{\tilde{X}_i\}} \sum q_i \tilde{X}_i c_i = \sum q_i X^*_i c_i = \sum q_i X^*_i (c_{1,i} - c_{2,i}) \\
= \sum q_i X^*_i c_{1,i} - \sum q_i X^*_i c_{2,i} < C_{1,\text{max}} - C_{2,\text{min}} \tag{C.3}
\]

Where, the last inequality follows from the fact that $X^*$ is the solution to LP in (C.3) and produces suboptimal results compared to $X^*_1$ and $X^*_2$ as in (C.1) and (C.2). Similarly, it can be shown that the lower bound derived from the convex hull, $C_{\text{min}}$, is greater than the lower bound obtained from the replicating portfolio, $C_{1,\text{min}} - C_{2,\text{max}}$. \qed
Table C.1: Tightness of the Bounds

<table>
<thead>
<tr>
<th>g</th>
<th>((C_{1,\text{max}} - C_{2,\text{min}}) - C_{\text{max}})/C_{\text{max}}</th>
<th>(C_{\text{min}} - (C_{1,\text{min}} - C_{2,\text{max}}))/C_{\text{min}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>3.1</td>
<td>5.5</td>
</tr>
<tr>
<td>1.3</td>
<td>7.6</td>
<td>13.9</td>
</tr>
<tr>
<td>1.5</td>
<td>10.6</td>
<td>20.1</td>
</tr>
<tr>
<td>1.7</td>
<td>12.8</td>
<td>24.6</td>
</tr>
<tr>
<td>1.9</td>
<td>14.1</td>
<td>25.4</td>
</tr>
<tr>
<td>2.1</td>
<td>14.7</td>
<td>23.9</td>
</tr>
<tr>
<td>2.3</td>
<td>13.4</td>
<td>17.7</td>
</tr>
<tr>
<td>2.5</td>
<td>10.9</td>
<td>12.6</td>
</tr>
<tr>
<td>2.7</td>
<td>7.3</td>
<td>8.2</td>
</tr>
<tr>
<td>2.9</td>
<td>4.1</td>
<td>4.4</td>
</tr>
<tr>
<td>3.0</td>
<td>2.7</td>
<td>2.8</td>
</tr>
</tbody>
</table>

We compare the tightness of the bounds obtained from the convex hull to those obtained from the replicating portfolio in the following numerical analysis. As in Section 4, we consider a reinsurance contract on the accumulated hurricane losses in the state of Florida, with a six month time to maturity and with a deductible and a ceiling equal to 2 and 5 billion dollars, respectively. The combined distribution of the hurricane landing and intensities are derived from the actual hurricane landing data in the state of Florida, as discussed in Section 5. We also assume that there is a futures contract on the same underlying loss process. The futures price is set equal to a multiple of the expected hurricane loss: \( F = gE[H] \). Table C.1 shows the percentage difference between the upper and lower bounds obtained from the convex hull and those found from the replicating portfolio.

As evident from the table, the bounds obtained from the convex hull method are considerably tighter than those obtained from the replicating portfolio, especially for the more realistic midrange values of \( g \).

D Proof of Lemma 2

The boundary of the convex hull of the conditional payoff one period prior to the contract expiration can be characterized by the following relations.\(^{26}\)

a) For \( V_{T-1} < \overline{V}_i \):

\[
C_{T-1,\text{upper}}(V_{T-1}, \overline{P}_i) = R^{-1}\left[ \overline{C}_{T-1} - \overline{C}_{T_0} - V_{T-1}(\overline{P}_i - \overline{P}_0) + V_{T-1} + \overline{C}_{T_0} \right], i \leq \overline{i};
\]

\[
C_{T-1,\text{upper}}(V_{T-1}, \overline{H}_i) = R^{-1}\left[ V_{T-1} + \overline{C}_T \right], i \in (i, \overline{i});
\]

\[
C_{T-1,\text{upper}}(V_{T-1}, \overline{H}_i) = R^{-1}\left[ \overline{V}_h - \overline{V}_i \right], i \geq \overline{i};
\]

\[
C_{T-1,\text{lower}}(V_{T-1}, \overline{P}_i) = R^{-1}(V_{T-1} + \overline{C}_T_1), i \leq \overline{i};
\]

\[
C_{T-1,\text{lower}}(V_{T-1}, \overline{H}_i) = R^{-1}\left[ V_{T-1} + \overline{C}_{T_1} + \frac{\overline{V}_h - \overline{V}_i - \overline{C}_{T_1} - V_{T-1} - \overline{C}_T}{H_n - \overline{H}_i} \right], i > \overline{i}.
\]

b) For \( \overline{V}_i \leq V_{T-1} < \overline{V}_h \) relations (3.2) yield now a graph \( \overline{C}_{T_1}(\overline{H}_i) \) that is concave. The

\(^{26}\)Note that in (D.1) the values \( \overline{i} \) and \( \overline{i} \) are non-increasing functions \( \overline{i}(V_{T-1}) \) and \( \overline{i}(V_{T-1}) \).
convex hull now becomes

\[
C_{T-1,i,\text{upper}}(V_{T-1}, \overline{H}_i) = R^{-1}(V_{T-1} + \overline{C}_T),
\]
\[
C_{T-1,i,\text{upper}}(V_{T-1}, \overline{H}_i) = R^{-1}[\overline{V}_h - \overline{V}_l],
\]
\[
C_{T-1,i,\text{lower}}(V_{T-1}, \overline{H}_i) = R^{-1}[(\overline{H}_i - \overline{H}_0) \overline{V}_h - \overline{V}_l - \overline{C}_T]_+ + V_{T-1} + \overline{C}_T.
\] (D.1b)

c) For \( V_{T-1} \geq \overline{V}_h \) the convex hull is clearly the line

\[
C_{T-1,i,\text{upper}}(V_{T-1}, \overline{H}_i) = C_{T-1,i,\text{lower}}(V_{T-1}, \overline{H}_i) = R^{-1}[\overline{V}_h - \overline{V}_l], \forall i.
\] (D.1c)

Proof: Conditional on \( V_{T-1} \), \( C_{T-1}(V_{T-1}|F_T) \) satisfies the market equilibrium conditions (2.4)-(2.5). Applying the transformations (2.6) to its payoff given by (3.2), we note that the admissible values of \( C_{T-1}(V_{T-1}, H_T) \) satisfy (2.7)-(2.8), implying that they must lie within the convex hull of the points \( \overline{C}_T \), whose form is now conditional on \( V_{T-1} \) as in (3.2). For case (a) the function \( \overline{C}_T(\overline{H}_i) \) has the same shape as in Lemma 1, with the convex hull being, therefore, given by (D.1a). For the case (b), \( \overline{C}_T(\overline{H}_i) \) is concave, while for (c) it is constant, thus proving the remaining part of the Lemma.

\[\Box\]

E Proof of Lemma 3

Given the value of the state variable \( V_{T-1} \), and value of the futures price \( F_{T-1} \), the two bounds can be found as the intersection of the vertical line stemming from \( F_{T-1} \) with the convex hull, whose boundaries are presented in Lemma 2:

\[
C_{T-1,\text{max}}(V_{T-1}|F_{T-1}) = R^{-1}C_{i,F,\text{upper}}, V_{T-1} \leq \overline{V}_{T-1},
\]
\[
C_{T-1,\text{max}}(V_{T-1}|F_{T-1}) = R^{-1}(\overline{V}_h - \overline{V}_l), V_{T-1} > \overline{V}_{T-1}
\]
\[
C_{T-1,\text{min}}(V_{T-1}|F_{T-1}) = R^{-1}C_{i,F,\text{lower}}, V_{T-1} < \overline{V}_{T-1},
\]
\[
C_{T-1,\text{min}}(V_{T-1}|F_{T-1}) = R^{-1}(\overline{V}_h - \overline{V}_l), V_{T-1} > \overline{V}_{T-1}
\] (E.1)

\[
C_{i,F,\text{upper}} \equiv \frac{\overline{H}_{i+1} - F_{T-1}}{\overline{H}_{i+1} - \overline{H}_i} C_{T-1,i,\text{upper}} + \frac{F_{T-1} - \overline{H}_{i}}{\overline{H}_{i+1} - \overline{H}_i} C_{T-1,i,\text{lower}},
\]
\[
C_{i,F,\text{lower}} \equiv \frac{\overline{H}_{i+1} - F_{T-1}}{\overline{H}_{i+1} - \overline{H}_i} C_{T-1,i,\text{lower}} + \frac{F_{T-1} - \overline{H}_{i}}{\overline{H}_{i+1} - \overline{H}_i} C_{T-1,i,\text{upper}},
\]
\[i^* : \overline{H}_{i^*} \leq F \leq \overline{H}_{i^*+1}\]

In the above expressions \( C_{T-1,i,\text{upper}} \) and \( C_{T-1,i,\text{lower}} \) are as defined in (D.1). Moreover, for a given \( F_{T-1} \) the boundary values \( \overline{V}_{T-1}, \overline{V}_{T-1}^1 \) and \( \overline{V}_{T-1}^2 \), that determine the break points of the concave functions \( C_{T-1,\text{max}}(V_{T-1}|F_{T-1}) \) and \( C_{T-1,\text{min}}(V_{T-1}|F_{T-1}) \), are defined by the values of \( V_{T-1} \) at which the intersection of the futures price with the convex hull given by (D.1) switches between two successive portions of the piecewise linear (convex) or concave (linear) functions of the upper (lower) boundaries of the convex hull.

24
Proof: The proof is similar to Proposition 1. As $V_{T-1}$ increases the upper bound is initially at some $i \leq \tilde{i}$, then at $i \in (\tilde{i}, h)$ and eventually at an $i > \tilde{i}$; this proves the two first equations of (E.1). The concavity of $C_{T-1,\max}(V_{T-1}|F_{T-1})$ follows directly from the concavity of the upper boundary of the convex hull for $i > \tilde{i}$. Similarly, for $C_{T-1,\min}(V_{T-1}|F_{T-1})$ the properties of the bound with respect to $V_{T-1}$ stem from the definition of $\tilde{i}(V_{T-1})$, with $\tilde{i}(V_{T-1}) = 0$ and $\tilde{i}(0) > \tilde{i}$, which imply that the lower bound lies also at some $i \in (\tilde{i}, h)$, thus yielding the piecewise linearity and concavity of $C_{T-1,i,\lower}(V_{T-1}, H_i)$ with respect to $V_{T-1}$.  

\section*{F Proof of Proposition 2}

We use induction and we rely on the concavity of the bounds.\footnote{If the bounds are neither convex nor concave we use the convexification described in Section 2 to derive the convex hull and then proceed as in proposition 1} At $T - 1$ the proposition holds by Lemma 3. Assume that it holds at $t + 1$. Then the bounds on $C_t(V_t|F_t)$ are found by the following program:

$$\begin{align*}
\max & \{\min_{\tilde{X}_i} \{R^{-1} \sum_0^n q_i \tilde{X}_i C_{t+1}(V_{t+1}|F)\}\}, \\
\min & \{\max_{\tilde{X}_i} \{R^{-1} \sum_0^n q_i \tilde{X}_i C_{t+1}(V_t + \kappa H_i|F)\}\} \quad \text{(F.1)}
\end{align*}$$

subject to the constraints in (3.3). Applying the transformation (2.6) and (3.4) to this program, we find that it is reduced to $\max \{\min_{\tilde{X}_i} \{R^{-1} \sum_0^n \tilde{X}_i \tilde{C}_{t+1,i}\}\}$ subject to (2.7). Since by the induction hypothesis $C_{t+1,\min}(V_{t+1}|F) \leq C_{t+1}(V_{t+1}|F) \leq C_{t+1,\max}(V_{t+1}|F)$, the following programs yield upper and lower bounds on $C_t(V_t|F)$, subject to (2.7):

$$\begin{align*}
\max & \{R^{-1} \sum_0^n \tilde{X}_i \tilde{C}_{t+1,\max,i}\}, \\
\min & \{R^{-1} \sum_0^n \tilde{X}_i \tilde{C}_{t+1,\min,i}\}\quad \text{(F.2)}
\end{align*}$$

Since both bounds at $t + 1$ are increasing and concave functions and the transformation (2.6)-(3.4) preserves the concavity of the functions $\tilde{C}_{t+1,\alpha,i}$, $\alpha = \max, \min, i = 0, \ldots, n$ with respect to $\tilde{X}_i$, $i = 0, \ldots, n$, the results of the programs (F.2) yield a value $C_{t,\max}(V_t|F)$ at the intersection of the futures price with the upper boundary of the convex hull of the points $(\tilde{C}_{t,\max,i}, H_i)$ as in (E.1). The value $C_{t,\min}(V_t|F)$, on the other hand, would lie on the lower boundary of the convex hull of the points $(\tilde{C}_{t,\min,i}, H_i)$, which is the straight line connecting the initial and final points, as in (E.1). Last, the concavity of both bounds follows from the fact that as $V_t$ increases both bounds trace the shape of the convex hull in its concave and linear parts respectively, which eventually degenerates into a straight line.  \[\square\]
References


26


Perrakis, S. (1988). Preference-free option prices when the stock return can go up, go down, or stay the same. *Advances in Futures and Options Research* 3.


Table 5.1: The Bounds and the Number of Periods

<table>
<thead>
<tr>
<th>N</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>Merton Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.60</td>
<td>2.75</td>
<td>1.22</td>
</tr>
<tr>
<td>2</td>
<td>3.47</td>
<td>3.59</td>
<td>1.62</td>
</tr>
<tr>
<td>3</td>
<td>3.70</td>
<td>3.81</td>
<td>1.77</td>
</tr>
<tr>
<td>4</td>
<td>3.80</td>
<td>3.91</td>
<td>1.85</td>
</tr>
<tr>
<td>5</td>
<td>3.86</td>
<td>3.97</td>
<td>1.89</td>
</tr>
<tr>
<td>6</td>
<td>3.89</td>
<td>4.00</td>
<td>1.93</td>
</tr>
<tr>
<td>7</td>
<td>3.92</td>
<td>4.02</td>
<td>1.95</td>
</tr>
<tr>
<td>8</td>
<td>3.93</td>
<td>4.04</td>
<td>1.97</td>
</tr>
</tbody>
</table>

This table shows the multi-period bounds and the Merton price for different numbers of periods. The bounds are calculated for a reinsurance contract with six months left to maturity, and with a deductible and ceiling equal to 1 and 10 billion dollars, respectively. The parameters are $R = 1.01$, $\lambda = 0.9$, $g = 2$, and $\kappa = 0.41$.

Table 5.2: Effect of CAT Event Risk Premium on the Bounds

<table>
<thead>
<tr>
<th>g</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>Merton Price</th>
<th>$g \cdot$ Merton Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.97</td>
<td>1.97</td>
<td>1.97</td>
<td>1.97</td>
</tr>
<tr>
<td>1.2</td>
<td>2.36</td>
<td>2.38</td>
<td>1.97</td>
<td>2.36</td>
</tr>
<tr>
<td>1.5</td>
<td>2.96</td>
<td>3.02</td>
<td>1.97</td>
<td>2.96</td>
</tr>
<tr>
<td>1.7</td>
<td>3.36</td>
<td>3.44</td>
<td>1.97</td>
<td>3.35</td>
</tr>
<tr>
<td>2</td>
<td>3.93</td>
<td>4.04</td>
<td>1.97</td>
<td>3.94</td>
</tr>
<tr>
<td>2.3</td>
<td>4.48</td>
<td>4.61</td>
<td>1.97</td>
<td>4.53</td>
</tr>
<tr>
<td>2.5</td>
<td>4.82</td>
<td>4.97</td>
<td>1.97</td>
<td>4.93</td>
</tr>
<tr>
<td>2.7</td>
<td>5.15</td>
<td>5.32</td>
<td>1.97</td>
<td>5.32</td>
</tr>
<tr>
<td>3</td>
<td>5.61</td>
<td>5.81</td>
<td>1.97</td>
<td>5.91</td>
</tr>
<tr>
<td>3.5</td>
<td>6.29</td>
<td>6.54</td>
<td>1.97</td>
<td>6.90</td>
</tr>
<tr>
<td>4</td>
<td>6.86</td>
<td>7.16</td>
<td>1.97</td>
<td>7.89</td>
</tr>
</tbody>
</table>

This table shows the multi-period bounds and the Merton price for different values of $g$. The bounds are calculated for a reinsurance contract with six months left to maturity, and with a deductible and ceiling of 1 and 10 billion dollars respectively. The parameters are $N = 8$, $R = 1.01$, $\lambda = 0.9$, and $\kappa = 0.41$.

Table 5.3: Effect of the Number of Recursions on the Tightness of the Bounds

<table>
<thead>
<tr>
<th>Number of Recursions</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.46</td>
<td>4.21</td>
</tr>
<tr>
<td>2</td>
<td>2.94</td>
<td>4.19</td>
</tr>
<tr>
<td>4</td>
<td>3.64</td>
<td>4.10</td>
</tr>
<tr>
<td>8</td>
<td>3.93</td>
<td>4.04</td>
</tr>
</tbody>
</table>

This table shows the values of the two bounds for different number of recursions. The bounds are calculated for reinsurance contract with six months left to maturity, and with a deductible and ceiling of 1 and 10 billion dollars respectively. The parameters are $N = 8$, $R = 1.01$, $\lambda = 0.9$, $g = 2$, and $\kappa = 0.41$. 

28
### Table 5.4: Multi-Period and Single-Period Bounds

<table>
<thead>
<tr>
<th>N</th>
<th>Single-Period Lower Bound</th>
<th>Single-Period Upper Bound</th>
<th>Multi-Period Lower Bound</th>
<th>Multi-Period Upper Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.60</td>
<td>2.75</td>
<td>2.60</td>
<td>2.75</td>
</tr>
<tr>
<td>2</td>
<td>3.47</td>
<td>3.59</td>
<td>3.39</td>
<td>3.67</td>
</tr>
<tr>
<td>3</td>
<td>3.70</td>
<td>3.81</td>
<td>3.20</td>
<td>3.90</td>
</tr>
<tr>
<td>4</td>
<td>3.80</td>
<td>3.91</td>
<td>2.89</td>
<td>4.02</td>
</tr>
<tr>
<td>5</td>
<td>3.86</td>
<td>3.97</td>
<td>2.72</td>
<td>4.09</td>
</tr>
<tr>
<td>6</td>
<td>3.89</td>
<td>4.00</td>
<td>2.60</td>
<td>4.15</td>
</tr>
<tr>
<td>7</td>
<td>3.92</td>
<td>4.02</td>
<td>2.52</td>
<td>4.19</td>
</tr>
<tr>
<td>8</td>
<td>3.93</td>
<td>4.04</td>
<td>2.46</td>
<td>4.21</td>
</tr>
</tbody>
</table>

This table compares the multi-period and single-period bounds for different numbers of periods. The bounds are calculated for a reinsurance contract with six months left to maturity, and with a deductible and ceiling equal to 1 and 10 billion dollars, respectively. The parameters are $R = 1.01$, $\lambda = 0.9$, $g = 2$, and $\kappa = 0.41$. 
Figure 5.1: Underlying Loss Process

This figure shows the discrete time underlying loss process for two periods. $V$ is the accumulated loss, $H$ is the intensity of the landed hurricane, and $\kappa$ is the multiplier translating the hurricane intensity into dollar losses.
Figure 5.2: Recursive Evolution of the Bounds

The multi-period lattice showing the recursive process for the upper bound starting from the maturity of the contract.

Figure 5.3: The Bounds and the Number of Periods

Multi-period bounds and the Merton’s price for different number of periods. The bounds are calculated for a reinsurance contract with six months left to maturity, and with a deductible and ceiling equal to 1 and 10 billion dollars, respectively. The parameters are $R = 1.01$, $\lambda = 0.9$, $g = 2$, and $\kappa = 0.41$. 
Figure 5.4: **Effect of CAT Event Risk Premium on the Bounds**

Multi-period bounds and Mertons price for different values of $g$ in the relation $F = g.E[H]$. The bounds are calculated for a reinsurance contract with six months left to maturity, and with a deductible and ceiling of 1 and 10 billion dollars respectively. The parameters are $N = 8$, $R = 1.01$, $\lambda = 0.9$, and $\kappa = 0.41$.

Figure 5.5: **Effect of the Number of Recursions on the Tightness of the Bounds**

The two bounds for different numbers of recursions. The bounds are calculated for a reinsurance contract with six months left to maturity, and with a deductible and ceiling of 1 and 10 billion dollars respectively. The parameters are $N = 8$, $R = 1.01$, $\lambda = 0.9$, $g = 2$, and $\kappa = 0.41$. 

32
Figure 5.6: Multi-Period and Single-Period Bounds

The figure compares the multi-period and single-period bounds for different numbers of periods. The bounds are calculated for a reinsurance contract with six months left to maturity, and with a deductible and ceiling equal to 1 and 5 billion dollars, respectively. The parameters are \( R = 1.01 \), \( \lambda = 0.9 \), \( g = 2 \), and \( \kappa = 0.41 \).